ME525 Applied Acoustics Lecture 4, Winter 2024

Peter H. Dahl

Complex harmonic exponential notation and linearization

Recall the linearization rule: ignore products of small acoustic variables (p_1, ρ_1, etc.), in the process of linearization, and also for corresponding spatial or temporal derivatives so long as the variable is continuous (or smooth) in space and time. This is nearly always the case with sound. Thus, both ξ and $\frac{\partial \xi}{\partial x}$, $\frac{\partial \xi}{\partial t}$, etc., are considered small, and products of such small variables are to be ignored in the process of forming a linearized approximation.

The easiest way to see that partial derivatives are also small quantities is to assume a harmonic time dependence in the acoustic variables. For example, take the acoustic displacement given by $\xi = \xi_0 e^{i(kx-\omega t)}$, where ξ_0 identifies displacement amplitude which can also be complex. Then $u = \frac{\partial \xi}{\partial t}$ equals $-i\omega\xi$. The factor $-i\omega$ is just another mulitiplicative constant, and the important magnitude information is contained in ξ, which we know is a small acoustic variable. Thus $\frac{\partial \xi}{\partial t}$ is considered "small" as well. Likewise, the spatial derivative $\frac{\partial \xi}{\partial x}$ equals $ik\xi$, and here the wavenumber k is the constant multiplying the small acoustic acoustic variable.

Develop good habits working with complex notation: the magnitude information in $-i\omega\xi_0 e^{i(kx-\omega t)}$ is completely defined by $\omega|\xi_0|$, as ω is just a simple real number. Never express the result as $i\omega|\xi_0|$, or $|i\omega\xi_0|$. The purely complex harmonic exponential $e^{i(kx-\omega t)}$ is magnitude 1 so this need not enter into the magnitude analysis, however there will be occasions where it is necessary to express a reduction of sound amplitude with increasing distance from the source due to sound attenuation (discussed later), for example, $e^{i(kx-\omega t)}e^{-\alpha x}$, where α represents an attenuation constant. The magnitude of this expression equals $e^{-\alpha x}$, and progressively reduces with increasing distance x.

Solution to the wave equation: Plane waves

A plane wave is so named because the phase fronts, describing the phase of pressure (or other acoustic variable) are in a plane. This is illustrated for a 1D case (Fig. 1) showing *plane wave propagation* down a tube. I've marked a phase front for positive (red) and negative (blue) pressure. A positive-to-positive (or negative-to-negative) distance is the *wavelength*, or λ. Take note: sound speed c, wavenumber k, sound frequency f radial frequency ω, and sound wavelength λ are all-important, and ubiquitous in the study of acoustic waves, with $f\lambda = c$ applies to all propagation (sea surface gravity waves, radio waves, etc.). You should memorize:

$$k = \frac{2\pi f}{c} = \frac{\omega}{c} = \frac{2\pi}{\lambda} \tag{1}$$

Assuming the tube goes on forever, a simple model for sound at single frequency f[1]is $p_1 = Ae^{i(kx-\omega t)}$ to representing a 1D, harmonic plane wave for pressure where A can in general be some complex amplitude. The 1D plane wave has a propagation direction depicted by the gray arrow with two

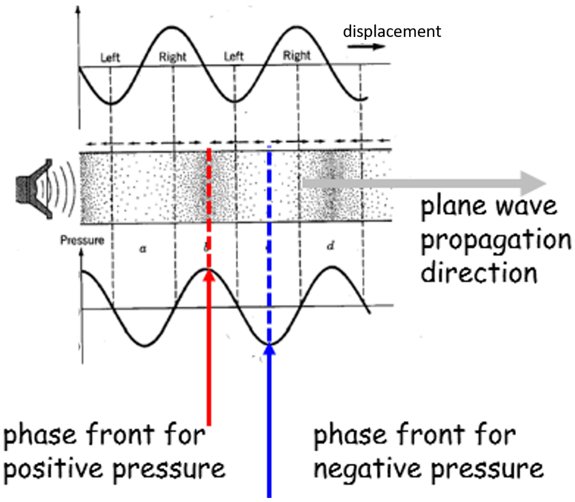

Figure 1: Plane wave propagation down a tube illustrating the phase fronts for positive (red) and negative (blue) pressure in the lower trace. The upper trace shows corresponding displacement (all values on relative scale), and it can be seen that for a plane wave, displacement is $180°$ out of phase with pressure.

phase fronts perpendicular to this propagation direction shown. This direction, perpendicular to the planar phase fronts, is called a ”ray”.

Consider next a somewhat idealized 3D plane wave (Fig. 2), where the red (positive) and blue (negative) planar surfaces represent surface of high and low acoustic pressure, respectively, which define the phase fronts. Only small sample (rectangular shape) of the surfaces are shown, as the phase fronts might ideally be of infinite extent. But practically there is some bound representing the finite spatial extent of the sound field, not unlike how a confined beam of light. The ray for this plane wave is pointing towards the right, slightly up.

To define the plane wave in 3D Cartesian coordinates, introduce polar angle α and azimuthal angle θ, with components of the wavenumber k defined as follows:

$$\begin{aligned} k_x &= k \sin\alpha \sin\theta \\ k_y &= k \cos\alpha \\ k_z &= k \sin\alpha \cos\theta \end{aligned} \tag{2}$$

where $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. The plane wave can be then expressed by

$$p_1(x, y, z, t) = Ae^{ik_x x + ik_y y + ik_z z} e^{-i\omega t} \tag{3}$$

[1]For this model to be accurate the frequency must satisfy $f < c/(1.71D)$ where D is the tube diameter. We’ll learn more about this later.

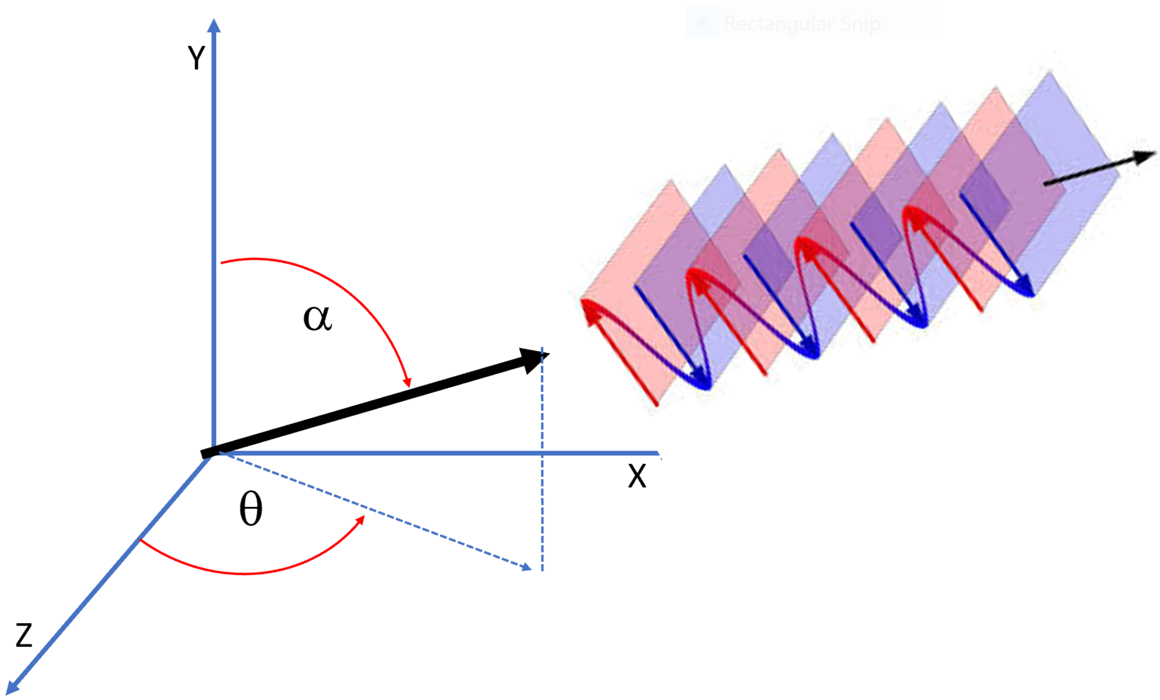

Figure 2: Right side: a plane wave depicted in a 3D sense with direction heading towards the right and slightly upward. The phase fronts of positive and negative phase (positive and negative pressure) are seen as planar surfaces. Left side: a coordinate system defining direction of the ray (black arrow) with polar angle α and azimuthal angle θ.

Convince yourself that Eq.(3) satisfies the wave equation developed in Lecture 3.

Plane waves and characteristic impedance of the acoustic medium $\rho_0 c$

We develop next one of the most important relations in acoustics that can only happen with plane waves–even though "true" plane waves are somewhat fictitious, a sound field under many conditions becomes very close to plane wave. Using the linearized Euler's equation [e.g. Eq.(6) from Lecture 3), take the gradient of Eq. (3) above to find

$$\begin{aligned} \rho_0 \frac{\partial u_x}{\partial t} &= -ik_x p_1 \\ \rho_0 \frac{\partial u_y}{\partial t} &= -ik_y p_1 \\ \rho_0 \frac{\partial u_z}{\partial t} &= -ik_z p_1. \end{aligned} \tag{4}$$

For example, the $x-$component of Eq.(4) reduces to $\rho_0 \omega u_x = k_x p_1$, and so on for the other two components. Thus, confirm: $(\rho_0\omega)^2 |\vec{u}| = k^2 |p_1|^2$, and therefore

$$|p_1|/|\vec{u}| = \rho_0 c \tag{5}$$

which is a basic property of plane waves. The new constant connecting acoustic pressure and velcoity is called the *characteristic impedance*, and is one of the most important quantities in acoustics.

Unfortunately, you will encounter a multiplicity of terms using impedance, some of which we will discuss subsequently, such as wave impedance and radiation impedance. There appears to be no getting around it, but at least the term *characteristic impedance* identified as $\rho_0 c$ is used fairly consistently within the field of acoustics. Stick with this terminology. In nearly every acoustic application you will want to know what $\rho_0 c$ is, for example, in analyzing the acoustic reflection between two media, say muscle and bone, or sea water and the seabed, the difference between the two characteristic impedances of the media will determine the strength of the reflection. The units of characteristic impedance are Pa times sec/m also known as a *Rayl* in honor of Lord Rayleigh.

At this point we are also making a change in notation for acoustic pressure p_1 having now a good understanding that this is first order, small acoustic variable. Given acoustic pressure is by the most important an easily observable the small acoustic variables, let us henceforth drop the subscript and use the symbol p for acoustic pressure.

Solution to the wave equation: Spherical waves

Plane waves were introduced first because of their inherent simplicity and that plane waves exhibit the property of pressure divided by velocity equals $\rho_0 c$. For real sources of sound, and particularly at ranges close to the source, plane waves are a poor approximation. However we show subsequently that for ranges r sufficiently far from the source of sound, such that $kr >> 1$, then phase fronts become *locally planar* and plane waves are not a bad approximation.

Let us now introduce spherical waves representing the simplest wave form in 3D that also describes well many physical applications in an exact manner. The pulsating sphere (Fig. 3) is a useful surrogate for sound source, such as human mouth, a loudspeaker, or an underwater acoustic transducer. We assert that the acoustic pressure from this source can be described as follows:

$$p(r,t) = \frac{A}{r} e^{ikr} e^{-i\omega t} \tag{6}$$

This expression depends on just one coordinate r equal to the radial distance from the center of origin of the sound source (Fig. 3) to the point where sound pressure field $p(r,t)$ is studied.

Does $p(r,t)$ satisfy the wave equation ($\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$)? An easy way to find out is see is to observe that for the $\nabla^2 p$, the Laplacian operator, in this spherically symmetric problem there is only $r-$ dependence without angular dependence. The Laplacian thus reduces to

$$\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial p}{\partial r}) \tag{7}$$

(See the resource section on website for handy summary of the Laplacian operator in different coordinate systems along with angular dependence.) Now break this out for the wave equation to include the time derivative as follows:

$$p_{rr} + \frac{2}{r} p_r - \frac{1}{c^2} p_{tt} = 0 \tag{8}$$

where p_r is short-cut notation for $\frac{\partial p}{\partial r}$, and similarly for p_{rr} and p_{tt}. Next a define a new variable equal to range times pressure, or rp. Using the new variable rp Eq. (8) simplifies to

$$(rp)_{rr} - \frac{1}{c^2} (rp)_{tt} = 0 \tag{9}$$

Observe Eq.(9) is the form of the wave equation in Cartesian coordinates which we have seen before, and is solved by functions of the form $f(r - ct)$. We thus find that $p = \frac{f(r-ct)}{r}$ which is an outgoing spherical wave, where pressure variation as a function of range goes as $\sim 1/r$.

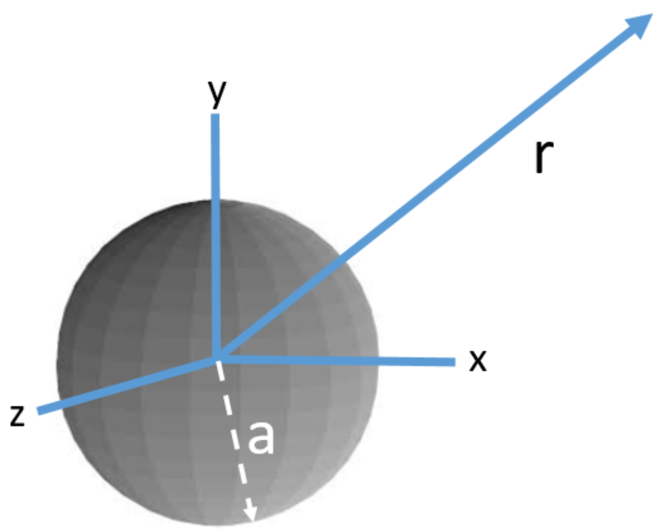

Figure 3: A spherical source of radius a placed at the center of the Cartesian coordinate system (x,y,z). Because of spherical symmetry, one spherical coordinate r describes all variation.

That sound pressure behaves in this manner is known as *spherical spreading*. You are experience spherical spreading of my voice in this classroom, or even better in environments without too much reflection such as on an outdoor sports field of grass–or even better–a field covered with snow (Fig. 4). For snow-covered field sound directed up into the air continues on and sound directed towards the snow is absorbed and effectively removed. Only the straight path connecting the source of sound to a receiver, as in your ears, counts.

Let's conclude by finding a specific expression for the constant A in Eq.(6) which will depend on what kind of boundary condition is imposed on the surface of the spherical source at $r = a$ (Fig. 3). This condition will ultimately determine how loud the sound is. There many ways to specify

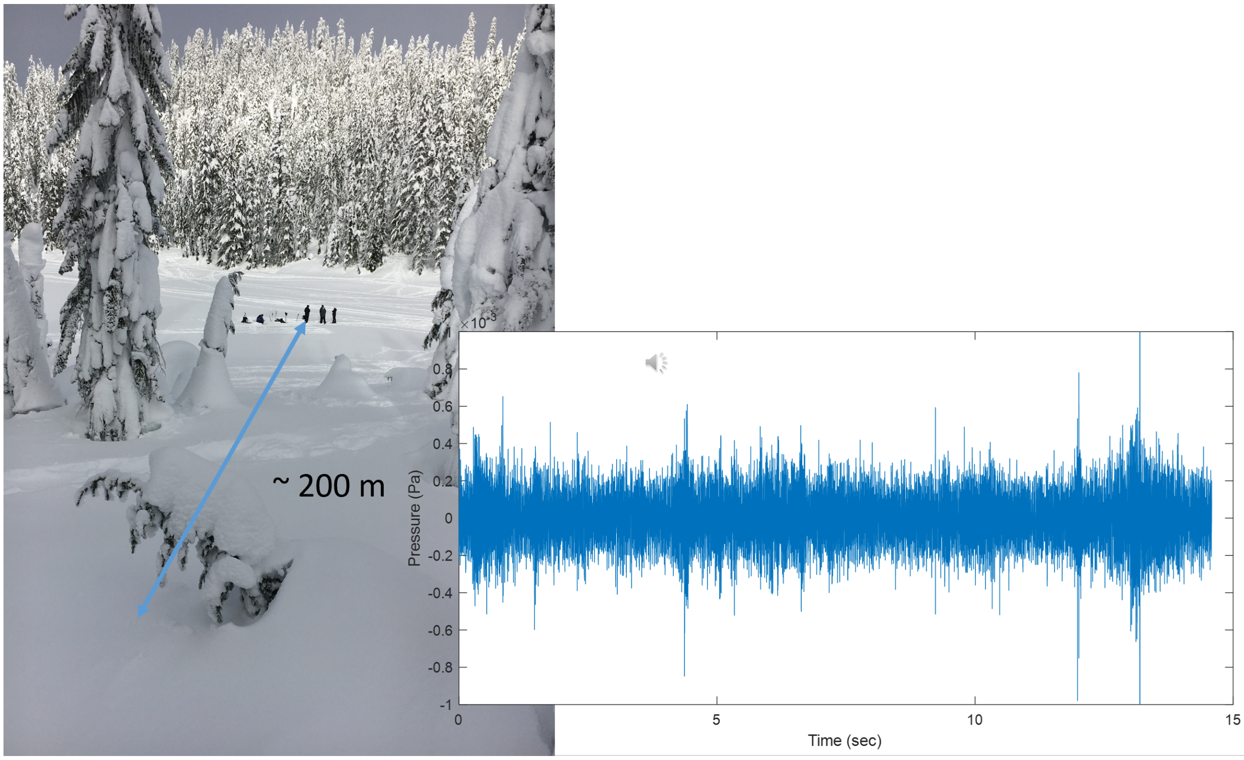

Figure 4: Snow covered field with group people approximately 200 m from iphone recording. The group's casual conversation can be easily heard. Spherical spreading represents a realistic, simple model for sound propagation from the group to the recording location.

this boundary condition; here we choose to specify the acoustic velocity on the spherical surface at $r = a$, as $u_0 e^{-i\omega t}$, where u_0 is a complex amplitude. Note: there is no physical necessity that this amplitude be complex, we can define it as real value e.g., 10^{-8}m/s while oscillating at some frequency as in 500 Hz. But leaving it complex provides more options such as specifying the phase of oscillation. For example, you might want two nearby sources to oscillate out of phase to achieve a noise cancellation effect.

First find a general relation between pressure gradient and velocity using Euler's equation as follows (remember we are now using just p in place of p_1):

$$i\omega\rho_0 u_r = \frac{\partial p}{\partial r} \tag{10}$$

where we now use u_r to denote velocity in the radial direction.

This leads directly to

$$u_r = \frac{A}{\rho_0 c}\frac{e^{ikr}}{r}(1 + \frac{i}{kr})e^{-i\omega t} \tag{11}$$

and using the given boundary condition $u_r(r=a) = u_0 e^{-i\omega t}$, identify the constant A as

$$A = \rho_0 c\ u_0 a (\frac{ka}{ka+i}) e^{-ika} \tag{12}$$

Observe the role the characteristic impedance $\rho_0 c$. Notice that the higher the $\rho_0 c$ the smaller the velocity amplitude necessary to achieve a given pressure amplitude. For example, think about the sphere operating in air (smaller $\rho_0 c$) versus water (larger $\rho_0 c$) . The final expression for pressure is

$$p(r,t) = a\rho_0 c\ \frac{u_0}{r} e^{ik(r-a)} (\frac{ka}{ka+i}) e^{-i\omega t} \tag{13}$$

This looks somewhat complicated, so it pays to first ask: is it dimensionally correct? Almost by inspection you can see that it is, knowing now that any ratio of acoustic pressure to acoustic velocity gives a quantity of dimension $\rho_0 c$ (this ratio may not be a purely real, with imaginary part, depending on circumstances, but it will always have this physical dimension of density times speed.)

Can you compute the root mean square (rms) value of $p(r,t)$ as function of range r? Take advantage of the complex exponential notation and simple harmonic time dependence!

ME525 Applied Acoustics Lecture 5, Winter 2024
more on near/far field, impedances, acoustic energy quantities

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Here's the solution for sound radiation from a sphere of radius a solved in the last lecture, based on the boundary condition for radial velocity (u_r), such that $u_r|_{r=a} = u_0 e^{-i\omega t}$.

$$p(r,t) = a\rho_0 c\,\frac{u_0}{r} e^{ik(r-a)} (\frac{ka}{ka+i}) e^{-i\omega t} \tag{1}$$

The non-dimensional parameters kr and ka

You will find many properties of the acoustic field that depend on range r and source length scale a; however a *much better understanding of these properties is always found upon assessing the non-dimensional parameters* kr and ka.

A good place to start is the ratio of pressure to velocity for wave under study as function of distance from source, either with data or with model. This is generically called the *specific acoustic impedance* (Junger and Feit, p. 31; Frisk p. 26). For the spherical wave in Eq.(1) the specific acoustic impedance is a function of range r is

$$\frac{p(r)}{u_r(r)} = \frac{\rho_0 c}{1 + i/kr} \tag{2}$$

where the time dependence $e^{-i\omega t}$ cancels.

Since kr is non-dimensional the specific acoustic impedance the spherical wave still has dimension $\rho_0 c$. Notice that the specific acoustic impedance in this case is complex, and dependent on kr or a combination of range from spherical source r and frequency via wavenumber k.

Observe now that when kr >>1 specific acoustic impedance approaches $\rho_0 c$ which is the *characteristic impedance*, representing the ratio of pressure to velocity for a plane wave. That a spherical wave behaves as a plane wave for $kr >> 1$ can be understood by considering the spherically expanding phase fronts becoming more like planar wave fronts for large distances away from the source. However this transition to plane wave behavior also depends on acoustic wavelength, and both wavelength and range are embodied by the important parameter kr.

For the limit $kr << 1$ convince yourself that in this case the specific acoustic impedance goes as $\sim -i\rho_0 ckr$, which further reduces to $-i\rho_0 \omega r$. Notice that in this limit, a fundamental property of sound propagation, the sound speed c, is no longer in effect and pressure and velocity are $90°$ out of phase. (The sign of this imaginary impedance term depends on the $e^{\pm i\omega t}$ convention is used, however there is no physical significance iin the sign.)

The two cases, $kr << 1$ and $kr >> 1$ are often referred to, respectively, as measurement positions r in the **near field** and **far field** of the source.

Next, examine the specific acoustic impedance on the surface of the sphere at $r = a$

$$\frac{p(a)}{u_r(a)} = \frac{\rho_0 c}{1 + i/ka} \tag{3}$$

which Junger and Feit call a *surface impedance.* Note if this quantity is multiplied by the area of radiating object, such as $4\pi a^2$ for the sphere, then instead of pressure over velocity the revised ratio is force over velocity. One motivation for this is that mechanical impedances are more traditionally expressed as a ratio of force to velocity amplitude and an impedance quantity of the same dimension is needed to be combined with a mechanical impedance for purposes of analyzing and modeling an entire system that involve driving mechanisms to generate sound.

Observe that for $ka >> 1$ the surface impedance of Eq.(3) becomes *real*, meaning acoustic pressure and radial velocity are in phase–when this happens (as seen later) there is efficient transmission of acoustic power away from the source. Conversely, when $ka << 1$ as in a very small sphere with respect to the acoustic wavelength, then acoustic pressure and radial velocity are out of phase, and there is inefficient transmission of acoustic power away from the source.

You experience this in overhearing high-pitched sounds from someone wearing small ear pods broadcasting music–you don't hear the low frequency bass because $ka << 1$, only the ear-pod person does, but you hear the high frequency "squeaky" sounds. The higher frequency (and thus k) puts $ka >> 1$ and efficient transmission of sound away from the source happens. Think also of a mosquito of characteristic dimension, a, flying close to your ear–now this sound source really has $ka << 1$–and as soon as it gets a few cm distant from your ear the sound appears to vanish. To summarize, much intuition emerges upon examining the characterize size of the source a with acoustic wavenumber k to understand the magnitude of ka.

The near/far field, and relation to acoustic energy

Acoustic velocity multiplied or scaled by $\rho_0 c$ provides the proper dimensional comparison with corresponding acoustic pressure. Using the generic equation for a spherical wave $p = \frac{A}{r} e^{i(kr - \omega t)}$ observe that for the $kr >> 1$ (or *far field*) $\rho_0 c v_r(t) \approx p(t)$, if not exactly being equal, where as for $kr << 1$ (or *near field*) these two acoustic quantities are out of balance in this regard. When $kr >> 1$ the **spherical wave** is behaving as **plane wave**, and $kr = 1$ represents an important transition point in this analysis.

For sound the relevant quantity is energy per unit volume, or energy density. The time-averaged kinetic energy per unit volume is

$$w_k = \frac{1}{2} \rho_0 u_{r_{rms}}^2, \tag{4}$$

where $u_{r_{rms}}$ is the root-mean-square (RMS) particle velocity in the radial r direction, again assuming a simple spherical wave.

Similarly the time-averaged potential energy per unit volume is

$$w_p = \frac{1}{2}\frac{1}{\rho_0 c^2}p_{rms}^2, \tag{5}$$

where p_{rms} is the RMS acoustic pressure. Verify yourself that w_k and w_p each have MKS dimension J/m^3.

Observe that both $u_{r_{rms}}$ and p_{rms} are necessarily real-valued quantities that enter into the energy calculations. For example, assume an expression for acoustic pressure for a harmonic spherical wave, $p(r,t) = \frac{A}{r}e^{ikr-i\omega t}$. The real part of this expression is required say representing a measurement, and for simplicity give the same name as $p(r,t)$ where

$$p(r,t) = \frac{|A|}{r}\cos(kr - \omega t + \phi_A). \tag{6}$$

This assums the amplitude is also complex as in $A = |A|e^{i\phi_A}$, however without any loss of generality we can set $\phi_A = 0$. The mean-square of this harmonic pressure is the time integral over one period T is

$$\frac{1}{T}\int_0^T p^2(r,t)dt = \frac{1}{2}\frac{|A|^2}{r^2} \tag{7}$$

giving the RMS pressure as $p_{rms} = \frac{1}{\sqrt{2}}\frac{|A|}{r}$. The analogous operation is performed to compute $u_{r_{rms}}$.

Observe that in this analysis where the acoustic variables are purely *harmonic*, as in $e^{-i\omega t}$, the RMS pressure and particle velocity are each a function of range r but no longer a function of time t given the time-integration over one period T. There are occasions where it makes sense to maintain a time dependence in these energy quantities,e.g., $w_k(t)$ and $w_p(t)$), representing, for example, the passage of ship for underwater noise or airplane for airborne noise.

It is useful to examine the kinetic and potential energy relation versus kr. Using again the expression for spherically symmetric wave the kinetic and potential energy densities (based on an arbitrary constant A) are plotted as function of kr (Fig. 1). Notice that for $kr << 1$ the kinetic energy exceeds the potential energy, for $kr >> 1$ they become equalized with transition point happening at the all-important value $kr \sim 1$. Thus, a very important transition range happens at $kr \sim 1$ identifying the approximate boundary between the *near field* and the *far field*.

More notes on decibel quantities

Decibels (dB) are used through out the world of acoustics. There are some specific uses, such as SPL or Sound Pressure Level as discussed in the homework. There are many other uses which we will define as the they come up in lectures. The primary use of decibel representation is to more

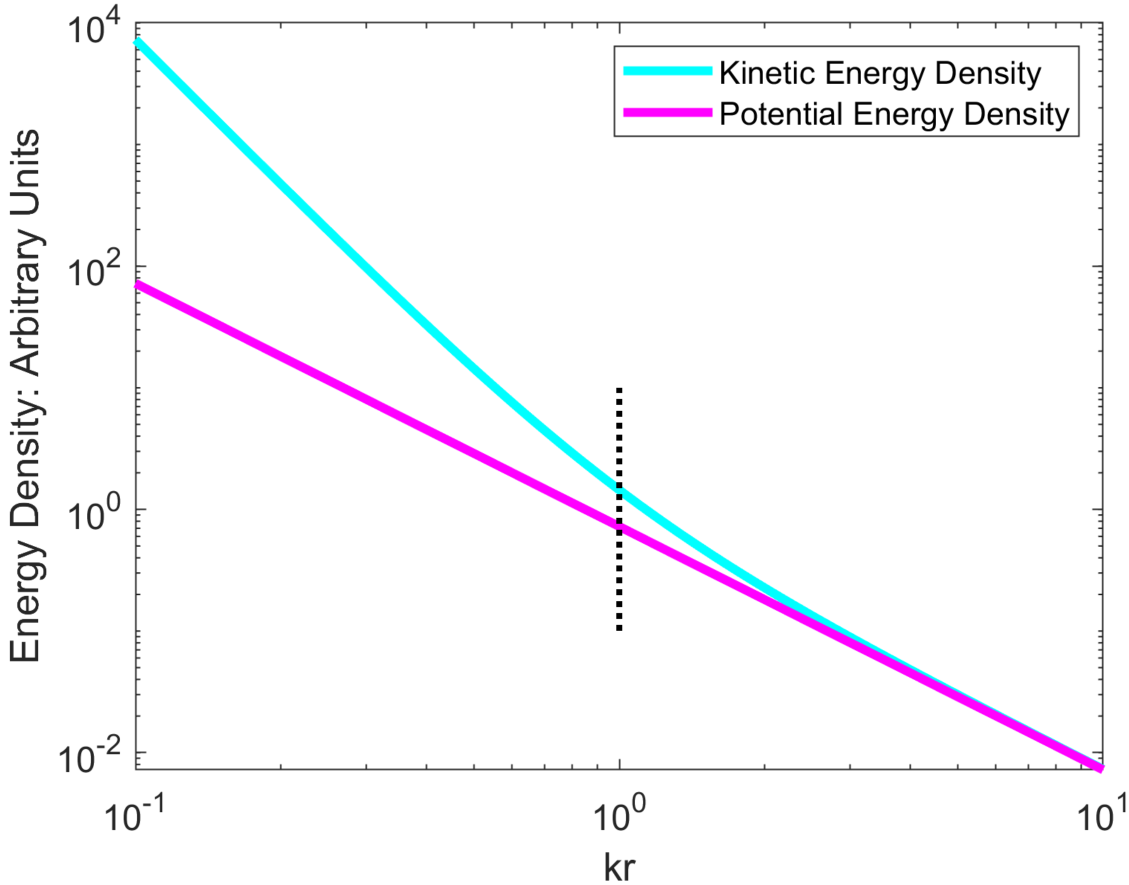

Figure 1: Relation between kinetic energy density w_{kin} and potential energy density w_{pot} versus kr for time-harmonic spherical wave in free-field conditions. Energy density units are arbitrary.

conveniently convey the large dynamic range of observations typically found in acoustics.

- Sound Pressure Level (SPL) is $20\log_{10}\frac{p_{rms}}{p_{ref}}$ where p_{ref} is 20μPa in air and 1μPa in water. It is good practice to identify the reference pressure p_{ref} when giving decibel value. For example, write: SPL equals 60 dB re 1 20μ Pa, where the "re" stands for the reference pressure. Upon knowing p_{ref} it is then easy to back out the value for p_{rms}, which evidently equals 0.02 Pa.

- The reference pressure for underwater sound equals 1 μ Pa. Thus, a SPL of 120 re 1μ Pa translates to p_{rms} equal to 1 Pa.

- Needless to say, you can't add decibels. The decibel values first need to be taken back to linear space; this is subtle, but not difficult. Your safest approach is to think of the decibel as $10\log_{10}$ of a squared quanity. For example, SPL (air) = $10\log_{10}(\frac{p_{rms}}{p_{ref}})^2$, and work with p_{rms}^2 or the mean-squared value. Now consider a case where the mean-squared value has doubled. Convince yourself that the increase equals 3 dB–worth committing to memory

- Another quite standard usage is the dBV, for ”dB voltage”, formed by $20 \log_{10} \frac{V_{rms}}{V_{ref}}$ where V_{ref} is 1 Volt rms. Note that voltage-squared is proportional to electrical power.

- For dB, never, ever, take the log of a complex or negative value!

References

Pierce, A. B, *Acoustics, An Introduction to its Physical Principals and Applications*, (Acoustical Society of America, and American Institute of Physics, 1989)
M. C. Junger and D. Feit,*Sound, Structures, and Their Interaction* (Acoustical Society of America, and American Institute of Physics, 1993)
F. Jacobsen and P. M. Juhl,*Fundamentals of General Linear Acoustics* (John Wiley & Sons, 2013).

ME525 Applied Acoustics Lecture 6, Winter 2024
The near and far field, Measurements by Jacobsen

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Getting the specific acoustic impedance for a spherical wave

Last time we discussed the ratio of pressure to velocity for wave under study as function of distance from source, either with data or with model, or specific acoustic impedance. For model based on a spherical, the ratio is

$$\frac{p(r)}{u_r(r)} = \frac{\rho_0 c}{1 + i/kr} \tag{1}$$

recalling that $\vec{u}$ in this case has only one radial component equal to u_r.

Given $p(r,t)$ how is the corresponding $u_r(r,t)$ found? Using Euler's equation confirm the following

$$u_r(r,t) = \frac{1}{i\omega\rho_0}\frac{\partial p(r,t)}{\partial r} \tag{2}$$

which assumes harmonic dependence $e^{-i\omega t}$ as in $p(r,t) = \frac{A}{r}e^{ikr-i\omega t}$, without further need to specify the pressure amplitude A. Now find

$$u_r(r,t) = \frac{1}{i\omega\rho_0}p(r,t)(ik - \frac{1}{r}) \tag{3}$$

which can be used in the pressure-velocity ration in Eq.(1) to yield the result for specific acoustic impedance.

Observe that unlike pressure $p(r,t)$, the radial velocity $u_r(r,t)$ for a spherical wave has two terms, the second going as $\sim \frac{1}{r}$ which ultimately vanishes for large r, which is more properly assessed in terms of the size of kr. Thus, in the near field ($kr << 1$) the first term of Eq.(3) dominates and the far field ($kr >> 1$) the second term dominates.

The Jacobsen measurements

Properties of near field and far field are further understood with the aird of some interesting measurements by Finn Jacobsen (1991). A pressure microphone and velocity probe are positioned a distance r from a loudspeaker that was broadcasting first near frequency 250 Hz, putting $kr << 1$ (Fig. 1) and appproximating a near-field condition. The exact range r for experiment is available from the Jacobsen study, but let us assume $r = 0.02$m or a little less than 1 inch. Is $kr << 1$?

Then frequency is increased to 1000 Hz, and assume now the range r is 0.02 m or about 8 inches away. This puts $kr >> 1$ (Fig. 2) and a far field condition. The figures refer to an "intensity probe" probably similar to the one shown in Fig. 3 of lecture 2, made by the Danish company Bruel & Kjær, and likely Jacobsen used a finite difference approximation to Euler's equation to first get acoustic acceleration in the radial direction, from which he could time integrate to obtain u_r (see the diagram of Fig. 3 from lecture 2).

To compare velocity on the same scale as a pressure , the velocity is multiplied, or scaled, by the characteristic impedance $\rho_0 c$, such that $u_r \rho_0 c$ and p have the same dimension. When $kr << 1$ the scaled-velocity is quite large relative to pressure, the two signals are 90° out of phase, and given the relative amplitude of velocity versus pressure, the kinetic energy density w_k will exceed the potential energy density w_p. In contrast, for $kr >> 1$ the pressure and scaled velocity are nearly equal (i.e., to the precision of these measurements), the two signals are now in phase, and we therefore we expect $w_k = w_p$.

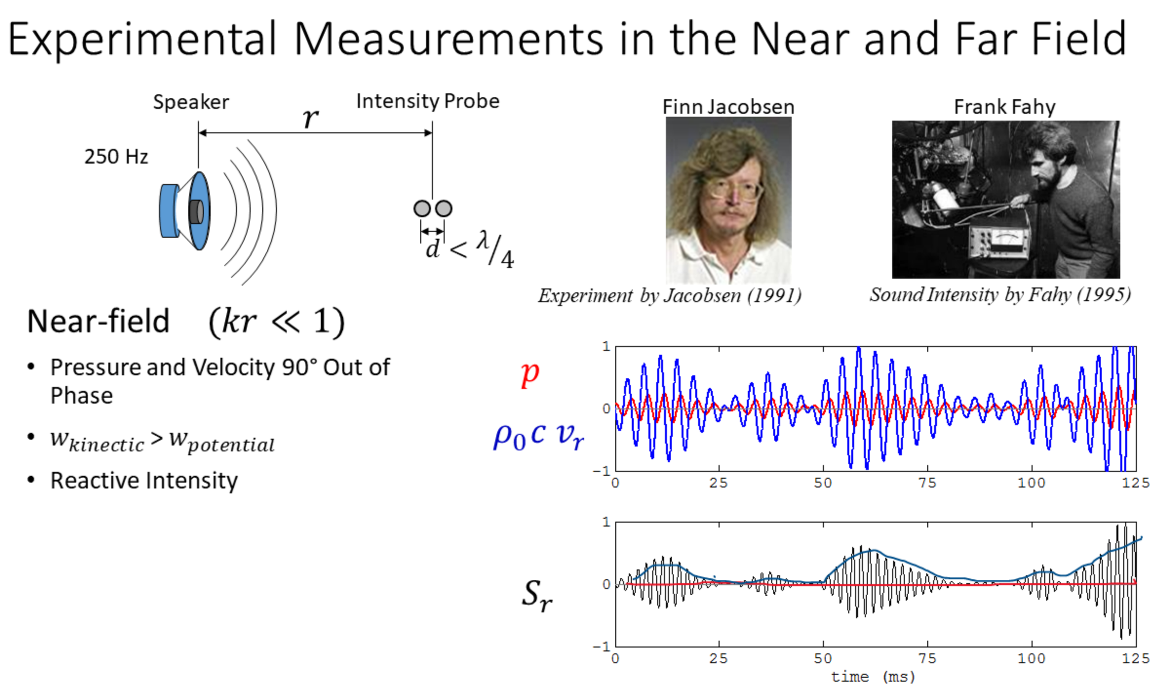

Figure 1: The Jacobsen measurements in the near field. Upper plot: $\rho_0 c$ times velocity considerably exceeds pressure, with these quantities $90°$ out of phase. Lower plot: corresponding Umov vector S_r (black), active intensity (red) and reactive intensity (blue)

The interpretation of this near field/far field business, beyond what is clearly indicated by the Eqs (1) and (3) evaluated for $kr << 1$ and $kr >> 1$, plus what we clearly observe in the Jacobsen data, can still be a bit mysterious. That is, what do we exactly mean (Lecture 5) when the specific acoustic impedance goes as $-i\rho_0 \omega r$ for $kr << 1$ in the near field? Recall the specific acoustic impedance lost its dependence on sound speed c, and we can write relation between pressure and velocity as

$$p(r,t) = -ir\rho_0 \omega u_r(r,t) = r\rho_0 a_r(r,t) \tag{4}$$

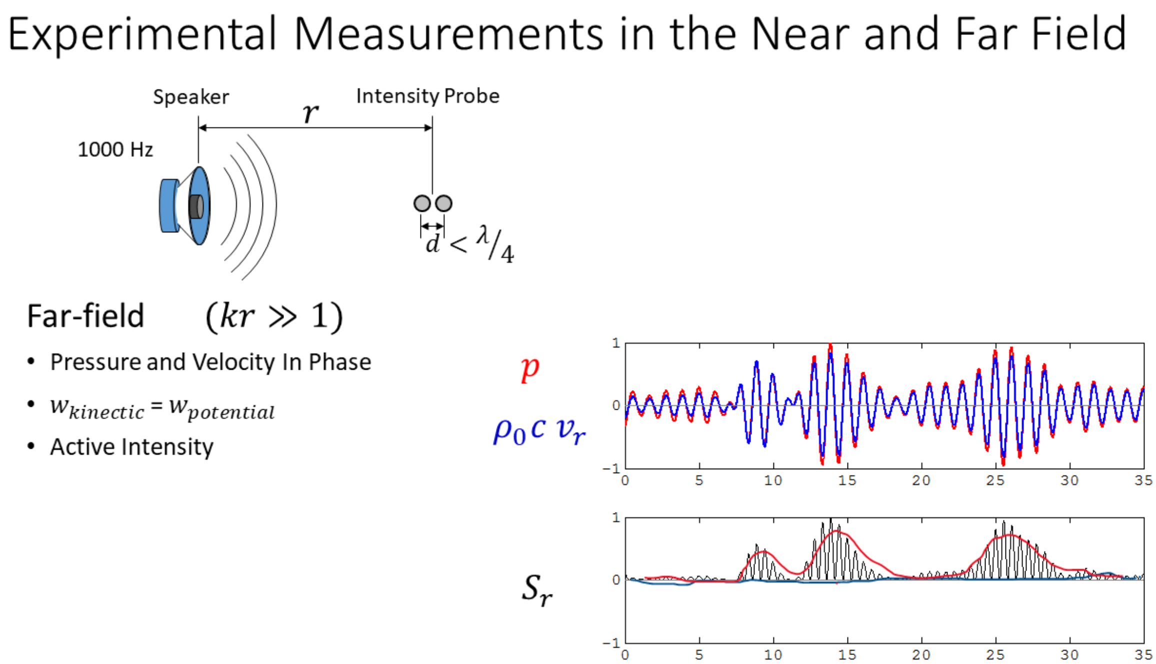

Figure 2: The Jacobsen measurements in the far field. Upper plot: $\rho_0 c$ times velocity equals pressure, with these quantities in phase. Lower plot: corresponding Umov vector S_r (black), active intensity (red) and reactive intensity (blue)

where $a_r(r,t)$ is acoustic acceleration. Dimensionally this a equivalent to a ”force equals mass times acceleration”, and process is the same as an accelerating parcel fluid subject to a force, hence the term *hydrodynamic near field* is sometimes used. So, what does this still mean?

I offer this (possibly crude) interpretation: (i) the sound speaker is activated or driven by a voltage signal producing movement (vibration) in the speaker element. Assume for simplicity the signal is harmonic at frequency f, (ii) the speaker vibrates and ”pushes” fluid away (air in this case) at this frequency, (iii) sound cannot be created instantly at the face of the speaker–the process has got to ramp up a bit spatially–giving some space such that the compressional part (involving c) can kick in. This happens at about $kr \sim 1$, or about $\frac{1}{6}$ of acoustic wavelength λ, where $\lambda = \frac{c}{f}$.

The near and far field, Umov vector, Active and Reactive Intensity

Sound intensity is a measure of the flow of acoustic energy within the sound field; formally intensity is a measure of the mean rate of energy flow in a unit area normal to the *direction* of sound propagation. Intensity is closely related, but not the same, as the sound kinetic and potential energy densities, since intensity is represented by the product of sound pressure times velocity and thus intensity has the dimension of energy per unit area per unit time or W/m^2. The term *energy flux density* is also synonymous with intensity. The Jacobsen data also help to illuminate the relation between pressure, velocity, and their combination giving intensity, as a function of kr.

We now introduce a new quantity: the energy flux density vector $\vec{S(t)}$, also called instantaneous intensity in W/m^2 which is plotted (Fig. 1, lower) for the case $kr << 1$. The vector $\vec{S(t)}$ is formed by the product of the pressure and velocity and since the fluid velocity in this geometry has just one component, u_r, means $S_r(t)$ has only one component in the radial direction. Important: $\vec{S(t)}$ is constructed by forming the product of pressure and velocity that are both *real* quantities, e.g., if modeling $\vec{S(t)}$ using, say $p(x,t) = \frac{A}{r} e^{ikr - i\omega t}$ then take the real part of this expression for the pressure contribution. The $S_r(t)$ is represented by the black line in the lower plot which also includes a sketch of the active (red line) and reactive (blue line) envelopes that we discuss later.

Different names exists, but in our research in vector acoustics, $\vec{S}$ is called the *Umov* vector. Observe by inspecting Fig. 1 that the *time average* of the Umov vector, denoted $\langle S_r(r,t) \rangle$, ought to be close to 0 if not exactly so. This is examplified by the red line in Fig. 1.

The interpretation is that instantaneous intensity in the form of $S_r(r,t)$ is flowing back and forth (or varying between positive and negative directions), rather than flowing in one direction, outward (or positive) from the source. This is an example of a *reactive* sound field, and represents another property of the acoustic near field. You hear the mosquito buzzing close to or within your ear–but a person a few feet away cannot hear that same mosquito because energy flux density vector $S(\vec{r}, t)$ associated with sound from mosquito was likely similar to the form shown in Fig. 1, with energy flowing back and forth rather than out and away from source.

In contrast, for the far field case with $kr >> 1$ (Fig. 2) the Umov vector no longer oscillates between positive and negative, and its time average is clearly non-zero (see again red line) , indicating that acoustic energy is flowing away from the source, and this is an example of a *active* sound field, representing another property of the acoustic far field.

The red lines in Figs. 1 and 2, are each a kind of "running time average" of the instantaneous intensity, or Umov vector, $S_{r,r}(t)$. Such if time average of instantaneous intensity is called **active intensity**. For the situation with $kr << 1$ (Fig. 1) active intensity in nearly 0, but for $kr >> 1$ (Fig. 2) active intensity is non-zero, and sound energy is flowing away from the source.

For the Jacobsen data think of active intensity as having one direction which is flowing outward and away from the source. In general, active intensity can have components in the x, y, z directions. An example is the data from an underwater explosive source, first shown as Fig. 5 in Lecture 1. The complete data including the Umov vector (Fig. 3) shows that $S_x(t)$ and $S_y(t)$ will have non-zero time averages, with $S_y(t)$ being larger because the bearing of this particular explosive source was in closer alignment with $y-$ axis of the sensor. The vertical component $S_z(t)$ oscillates more about 0, which is typical of the vertical component of underwater sound.

Returning now to the Jacobsen data, observe next a blue line that tends to follow the envelope of the oscillatory $S_r(r,t)$ for the case $kr << 1$ (Fig. 1). The is called **reactive intensity** which might also think of as a running average of the amount of intensity that is flowing back and forth. How this is computed will be discussed later. For the $kr << 1$ situation nearly 100% of the instantaneous intensity can be viewed as reactive intensity, which is consistent with nearly 0 active intensity is. In contrast reactive intensity is nearly 0 for the far field case $kr << 1$ (Fig. 2), and this is consistent with $S_r(r,t)$ being none-oscillatory and generally going in one direction outward.

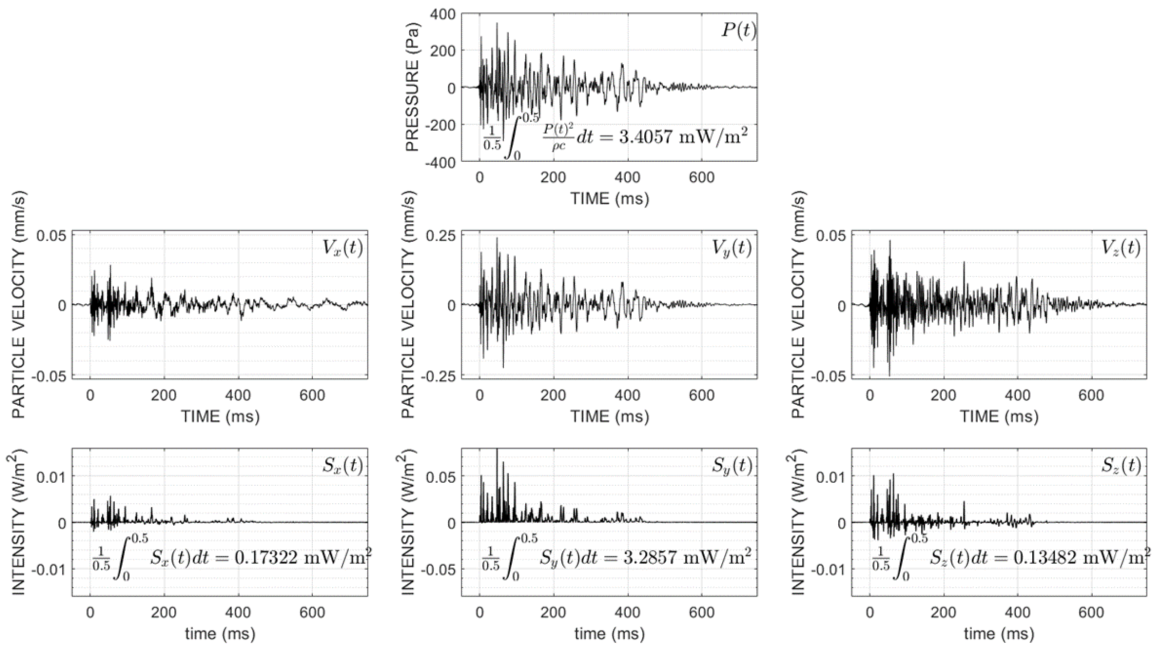

Figure 3: Acoustic pressure (top) and three components of acoustic velocity (middle) and Umov vector (bottom) of the underwater sound from an explosive source (31 g of TNT), made at range 10 km from the source in waters 75 m deep. Figure from Dahl and Dall'Osto (2019)

Let's look more closely from perspective of simple models. First, note that $\vec{S}(t) = p(t)\vec{u}(t)$ where these are real-valued quantities, say as recorded by Jacobsen. However, for analytic expressions, or models, of pressure and velocity that can be complex, we take the real-part of pressure and velocity as follows

$$\vec{S}(r,t) = Re\{p(r,t)\}Re\{\vec{u}(r,t)\} \tag{5}$$

For example, take simple generic expression for sound radiation from a sphere of radius a driven harmonically with time dependence $e^{-i\omega t}$,

$$p(r,t) = \frac{A}{r}e^{ikr-i\omega t} \tag{6}$$

For the following we don't need exact details A other than to assume it can be in general complex. So, write as $A = |A|e^{i\phi_A}$ where ϕ_A is the phase angle for this complex variable A. Use Eq.(3) for the necessary expression for radial velocity $u_r(r,t)$

Take real parts of each and multiply them to yield

$$S_r(r,t) = \frac{|A|^2}{r^2\rho_0 c}\{\cos^2(kr-\omega t+\phi_A) - \frac{\cos(kr-\omega t+\phi_A)\sin(kr-\omega t+\phi_A)}{kr}\} \quad (7)$$

With this expression we understand some key features of the Jacobsen data, specifically at $kr >> 1$ the first term dominates and $\cos^2$ behavior is expected as roughly shown by the data (although there is amplitude modulation in the Jacobsen data which we would need to incorporate in the form of $|A(t)|^2$). Similarly, at $kr << 1$ the second term dominates and the behavior is characterized by energy flow goes back and forth as described by the $\cos$ times a $\sin$, although again with amplitude modulation in the data. While both terms are equal in magnitude at $kr = 1$, representing a transition point, between the near and far field–which is a good rule of thumb.

Another key feature of Eq.(7) is that for $kr >> 1$, where the first term then dominates, then the radial time-varying intensity $S_r(r,t)$ for this spherical wave goes as $\frac{1}{r^2}$. Recall that for a spherical wave, the pressure amplitude goes as $\frac{1}{r}$. Thus $\frac{1}{r}$ dependence for pressure and $\frac{1}{r^2}$ dependence for intensity, are very important characteristics of spherical waves. Exceptions to this, e.g., underwater sound where the sound can be trapped between the sea surface and seabed and sound propagates in a *waveguide*, will be discussed later.

References

F. Fahy, *Foundations of Engineering Acoustics* (Elsevier Academic Press, San Diego, CA, 2001)
F. Jacobsen: "A note on instantaneous and time-averaged active and reactive intensity." *J. of Sound and Vibration* 147, 1991.
L. M. Brekhovskikh,*Waves in Layered Media* (Academic Press, New York, 1960)
R. B. Heyser, "Instantaneous Intensity" *Proceedings of the 81st Convention of the Audio Engineering Society* preprint 2399 (1986)
F. Jacobsen and P. M. Juhl,*Fundamentals of General Linear Acoustics* (John Wiley & Sons, 2013).
D. R. Dall'Osto, P. H. Dahl, and J. W. Choi, "Properties of the acoustic intensity vector field in a shallow water waveguide," *J. Acoust. Soc. Am.* 131, March 2012.